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of the

Linear Covariants of Ternary Forms.

Dissertation.

Submitted to the Board of University Studies of the Johns Hopkins University in Conformity with the requirements for the degree Doctor of Philosophy.

By.

Josiah Wesley Gain.

1914

169,095

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Introduction.

My purpose in this paper is to set forth a method of calculating the linear Covariants of ternary forms, and to determine the quadratic syzygies which they must satisfy. The work includes the Calculation of the Covariants of all sets of Conics and of two and three cubics.

In the case of two and three Conics and two cubics we have found what relations the Coefficients of these forms must satisfy in order that they may be linear Covariants.

Note Two Covariants of two Cubics and two syzygies of three Cubics have also been found.

In my work I have
found certain Egyptian connecting
the @ mathematic. The computation
of this fact I consider novel.
The method is using the
annihilators of the unit

By linear Combinants is meant those Combinants that are linear in the Coefficients and are essential part of my work is to express the Combinants in terms of these Coefficients.

Combinants in general and in particular linear Combinants are determined by their End coefficients which are annihilated by certain operators to be defined later.

symbolic Forms for Combinants
of any two ternary forms of
the same degree.

By Clebsch's Uebertragungs-
princip we pass from the symbolic forms
of Combinants of 2 binary forms
to the symbolic forms
of Combinants of 2 ternary
of the same degree as
the binary forms.

Every rational Invariant
of 2 binary forms of
the same degree is
a Combinant.
Let f_1, f_2 be the binary
forms

Then by a fundamental theorem on transvectants

$$(\xi_1 f_1 + \xi_2 f_2)^2 \,(\eta_1 f_1 + \eta_2 f_2)^2$$

$$= \xi_1 \eta_1 (f_1 f_1)^2 + \xi_1 \eta_2 (f_1 f_2)^2 + \xi_2 \eta_1 (f_2 f_1)^2 - (f_1 f_2)^2$$

$$= (\xi_1 \eta_2 - \xi_2 \eta_1)(f_1 f_2)^2 \quad \therefore \; f_1 f_2 \text{ is a covariant with } 2 \text{ variables.}$$

Therefore, to find the <u>linear</u> combinations of n binary k-ics, $a x^k$, $b x^k$, we first find the order of invariants of the two binary forms $a x^k$, $b x^k$ with k and $(k-1)$ by taking n equal to the number of variables to start with.

In this way we get the following linear Covariants

For Conics, $|abu|(ax)(bx)$

" Cubics, $|abu|(ax)^2(bx)^2, |abu|^2$

" Quartics, $|abu|(ax)^3(bx)^3, |abu|^2(ax)(bx)$

" Quintics, $|abu|(ax)^4(bx)^4, |abu|^2(ax)^2(bx)^2, |abu|^3$

And so on for higher forms.

Notation

For convenience in calculation the following notation shall be understood to hold unless clearly to the contrary.

The Conic would be written in the form:

$$a_{200}x_0^2 + a_{020}x_1^2 + a_{002}x_2^2 + 2a_{011}x_1x_2 + 2a_{101}x_0x_2 + 2a_{110}x_0x_1$$

A second and third Conic would be similarly written but with b and c instead of a.

In the matrix notation we have occasion to represent the Conic A by its determinant a similar notation holds

would be

$$a_{300}x_0^3 + a_{000}x_1^3 + a_{003}x_2^3 + 3a_{210}x_0^2x_1 + 3a_{201}x_0^2x_2 + 3a_{120}x_0x_1^2 + 3a_{021}x_1^2x_2 + 3a_{102}x_0x_2^2 + 3a_{012}x_1x_2^2 + 6a_{111}x_0x_1x_2.$$

The suffixes in order denoting the number of times the corresponding variable occurs. Thus a_{300} shows that the first variable occurs three times, the second and third not occurring at all, while a_{000} shows there is only one variable present, the second, which occurs three times.

Further, a form written as $|a\ b|$ shall be understood to stand for the determinant $\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}$ or $a_0b_1 - a_1b_0$.

$|abc|$ shall be understood to see the symbol for

$$\begin{vmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

A further notation, that of writing a line above the form, as $\overline{300}$, is a symbol for the determinant whose $\overline{300}.\overline{030}\ \overline{003}$ is a symbol for the determinant:

$$\begin{vmatrix} a_{300} & a_{030} & a_{003} \\ b_{300} & b_{030} & b_{003} \\ c_{300} & c_{030} & c_{003} \end{vmatrix}$$

A form written in brackets is a symbol for a true linear product

$(ax)_1$ for example, is $(a_0x_0 + a_1x_1 + a_2x_2)$ and $(ax)^2a_3$ means

$(a_0x_0 + a_1x_1 + a_2x_2)(a_0x_0 + a_1x_1 + a_2x_2)a_3$

Weight

I Point Coordinates.

A term containing k point coordinates only, e.g. $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k}$, has a weight $\alpha_1 + \alpha_2 + \dots + \alpha_k$, thus $x_1^{\alpha_1} x_2^{\alpha_2}$ has the weight 6.31; $x_1^{\alpha_1} x_2^{\alpha_2}$ has the weight 6.20. The figures in brackets showing how many times the corresponding variable occurs. Seeing the weight 1,11 would indicate the term $x_7 x_8 x_{11}$.

5 Line Coordinates

For line Coordinates the weight of ξ_0^3, ξ_1^3, ξ_2^3 are 0, 1, 10, 110 respectively; thus ξ_{01}^2 has the weight 0, 2, 2; ξ_{31}^2 has the weight 0, 0, 2; ξ_{23}^2 has the weight 2, 2, 2 and so on.

II Mixed Coordinates

In the case of mixed coordinates, that is transformations connecting of both point and line coordinates the weight of a term shall be the sum of the weights of the point and line coordinates. For example, $x_0^1 \xi_1^2$ having respectively the weights 0, 0, 0 and 0, 1, 1 the weight of this term is 0, 1, 1.

We must further that the equation of a line connecting the points x_0, x_1, x_2 and y_0, y_1, y_2 is

$$\begin{vmatrix} \bar{X}_0 & \bar{X}_1 & \bar{X}_2 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{vmatrix}$$

or $(x_1 y_2 - x_2 y_1)\bar{X}_0 + (x_2 y_0 - x_0 y_2)\bar{X}_1 + (x_0 y_1 - x_1 y_0)\bar{X}_2 = 0.$

Comparing this with $\bar{z}_0 X_0 + \bar{z}_1 X_1 + \bar{z}_2 X_2$ we see that $\bar{z}_0$, $\bar{z}_1$, $\bar{z}_2$ are homogeneous functions of $x_1 y_2 - x_2 y_1$, $x_2 y_0 - x_0 y_2$, $x_0 y_1 - x_1 y_0$ respectively.

So that in general we may take

$$\bar{z}_0 = x_1 y_2 - x_2 y_1,$$
$$\bar{z}_1 = x_2 y_0 - x_0 y_2,$$
$$\bar{z}_2 = x_0 y_1 - x_1 y_0.$$

This is of theoretical importance in the permutations we shall have both point and line coordinates.

IV The weight of Coefficients

The determinants that show the coefficients are called. A term consisting of the determinants $|01.200|$ will be said to have the weight $1,0,1 + 2.0.0$ or $3.0.1$

The use of Covariants

Throughout this paper it must be borne in mind that Covariants may have three different forms. They may be Covariants Contravariants, or mixed forms. The Covariants Consisting of Point Coordinates only, the Contravariants Consisting of both line Coordinates Only, Once the mixed Forms or Point Coordinates which are Made up of both Point and line Coordinates.

To calculate these terms we use what Blair Calls Annihilators. After finding a leading coefficient we use a differential operator successively upon this term and get the other terms of the Complementary.

Altogether there are six of these Annihilators - D^2, D, $D-1$, D^2-2D+1, D^2-1, D^2+1. The first five of these we observe to Blair's description of Annihilators. Since my Complements are made of determinants my Annihilators are to be understood as showing

For the determinants
I shall call the invariants
which I have, $I_2, I_3, I_4, \ldots I_n$.

I so for example, is the
algorithm

$$a_0 I_3 - 2 a_1 I_2 + 3 a_2 I_1 + \ldots$$

And a like relation
for the other. Cayley and C. Elliot
show in their theory of Quantics
p. 126, that the annihilator
of an invariant of a
system of quantics is
the sum of the annihilator
of each invariant of each
of the quantics of the
coefficients of the combinant
are made up of determinants
which are not of the effect

2 Inflorescence showing the
Lateral Form

Repeating on the seconds
Columns of the Determinant
which make up
the coefficients of the
Determinant in the second
What we get by expanding
the first Column by one and leaving
the second Column by the same
amount, leaving all the other
Columns unchanged, and then
performing the second
Operation upon all the
other Columns.

We leave the third Column
by one and leave the
third, We leave the third
and multiply the first and
a Number was Being for

with the other operators, the first
block is concerned with the Cross
Correlation Block and the second
is Elementary.

Suppose we have found
that $12\bar{0} \cdot 2\bar{1}0$ which is the Symbolic
for $\begin{vmatrix} a_1 & a_0 \\ b_1 & b_0 \end{vmatrix}$ is our Best Estimate

of a Resultant Line, the Coefficient
of V_0 and that we wish to
find the Coefficient of V_1.
Remove $\frac{1}{2}a_0$ between the first
by one rule we alternate with b_1
and the Coefficient of V_1

Now operating with
one the first the Intermediate
line will be recorded
$0\,3\bar{0} \cdot 2\bar{1}0$ - Eliminating $\frac{1}{2}a_0$

Same way for the determinant determinant is not zero 2,120.120| which means since the determinant has two rows alike

The Coefficients of x^2 h_0 Here 050.210.2121 with the second row we wanted zeros replaced with $L_{2,0}$ This operation happens the effect of leaving the first row. Multiplying the third we wanted get zero by operating the first row. Operating on the second Column with the Zeros now we get 2,050.111| as the coefficient of the term $h_0^2 h_2$.

We can say, then, that the operating on the coefficients of the terms x_0^1, x_0^2, x_0^3 gives the coefficients of the terms $x_0^{n+1}, x_0^{n+2}, x_0^{n+3}$. A procedure resembling finite difference operators.

Annihilators for Exact Coefficients

If without loss we can [illegible] Coefficients with the [illegible] Coefficients is [illegible] by step 2. For we can find the annihilator of the product $0 \subset 00\bar{3}\cdot 03\bar{2}$ vanishes.

$00\bar{3}\cdot c3\bar{2}$ is [illegible] to be annihilated by $x^2 = 0$.

We [illegible] if it is [illegible]

This vanishing due to the operation of certain Σ_2 by the [illegible] of the Exact Coefficients of Symmetric Powers [illegible]

mixed powers.
We mean the Leading Coefficient of Covariants of the type
by mean four Covariants of that Leading Coefficients of the type 4 or Leaving four Covariants also.
Finding the Leading Coefficients of Contravariants Leaving four Covariants Invariantly Independent from those of the Covariants.
The Leading Coefficients of mixed tensors have only two Covariants.
So knowing the weight of the Leading Coefficient we use Lemma, and Leaving it to determine whether

it has the form annihilator.
We can at once tell if
we have a Commutative,
Commutative and
Matrix form.

In the next page I have
given the examples
of these various forms.
By a glance at this
matrix we see at once
that if we find a Complex
which is annihilated by
the Sea the Sea
it is a Commutative Quaternion
we notice also that
from the Complex unit of
by raising to 3rd power.

Had we found the coefficient to be annihilated by $\Omega_{be'}$ and $\Omega_{cb'}$ only, our combinants would have been a mixed form; this coefficient belongs to the term λ_0 carried to some power.

Index Annihilators of the Leading Coefficients of Covariants Contravariants and Mixed Forms.

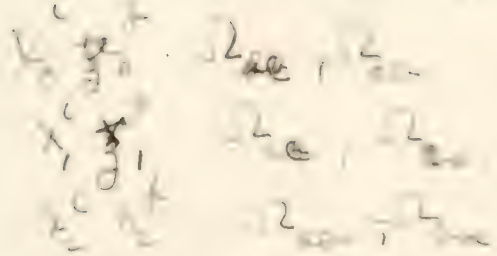
Covariants. Contravariants

$X_0^n \cdot \Omega_{be'} \Omega_{ce'} \Omega_{ab'} \Omega_{ca'}$ $\quad$ $Y_0^n \cdot \Omega_{ab'} \Omega_{ac'} \Omega_{ba'} \Omega_{cb'}$.

$X_1^n \cdot \Omega_{ac'} \Omega_{ca'} \Omega_{ab'} \Omega_{bc'} \ldots$ $\quad$ $Y_1^n \cdot \Omega_{ac'} \Omega_{ca'} \Omega_{ba'} \Omega_{bc'}$.

$X_{n-1}^n \cdot \Omega_{ab'} \Omega_{ba'} \Omega_{ac'} \Omega_{bc'}$ $\quad$ $Y_{n-1}^n \cdot \Omega_{ab'} \Omega_{ac'} \Omega_{ca'} \Omega_{cb'}$.

Mixed forms.



I shall deal later with the method of determining the order and class of these forms. To be able to tell to which particular class a combinant belongs - whether it is a Covariant, Contravariant or mixed form - is sufficient for our present purpose, since in the case of two Conics, two Cubics and some of the other forms of lower degree

we are able to find the Orders and Class from the Symbolic Formula. Then we Come to deal with the Combinants of three, and four Cubics, where we Cannot rely upon the Principle of Translation to give us all the Combinants this Method of determining the Orders and Class will be brought out.

The method of determining the objects to be used.

I. Covariant forms

Where we proceed by the method already shown in determining what particular forms we have (that is, whether Covariant, Contravariant or mixed forms) and when we determine what Ends particular we have, we then use Elliott's Symbol for Aronhold, considering that each particular form has a different Separatrix for each Invariant which also Varies with

the coefficient which we have selected to be our 2nd coefficient

This can be made easily known by a few illustrations.

Let us suppose that our combinant to a covariant C_{mn} and that by the above method we have found the coefficients $\lambda_{0,1,2}$ we find the q_{mn} $\lambda_0 C^{10} + \lambda_1 (C^{9,1} + C^{8,2}) + \lambda_2 (C^{7,3} + C^{6,4} + C^{5,5})$ or since C is a covariant it would be the same relation $\lambda_0 c^{\frac{1}{10}} + \lambda_1 (c^{\frac{2}{10}} + c^{\frac{1}{10}}) + \lambda_2 (c^{\frac{3}{10}} + c^{\frac{2}{10}} + c^{\frac{1}{10}})$ observing

Our J where J is our Feeding
coefficient

Substituting this
formula we get.

$$\xi_0^2 J + x_1 x_1 \frac{1}{2!} J - x_1 x_2 \frac{1}{2!} J + \frac{1}{2} x_1^2 \frac{1}{2!} - x_1 x_2 \frac{1}{2!} x_2^2 + \frac{1}{2} x_2^2 \frac{1}{2!} x_2^2.$$

By a glance at this expression we can tell just what alterations to use to find any terms of the coordinate.

We see that if the term is $x_0 x_1$ + it must be operated with Ω_{x_0} on J, if $x_0 x_2$ be the term we are looking for we operate with Ω_{x_0} on J and proceed similarly to find the remaining terms.

Had we found the
coefficient of u_1^2 the expansion
would have differed.

We would have
expanded $\lambda_1^2 \left(e^{\frac{1}{2}\lambda_2 a^2 + \frac{1}{2}\lambda_2 a^2} \right) S$ or
in full, in our notation,

$$\lambda_1^2 S + \lambda_1 \lambda_2 \Omega_{ba} S + \lambda_1 \lambda_2 \Omega_{ba} S + \frac{1}{2} \lambda_2^2 \Omega_{ba}^2 S +$$
$$+ \lambda_1 \lambda_2 \Omega_{bc} \Omega_{ba} S + \frac{1}{2} \lambda_2^2 \Omega_{ba}^2 .$$

Had we found the
coefficient of λ_2^2 we would
similarly have found that
that our leading
coefficient. Our expansion
in that case being
that of $\lambda_2^2 \left(e^{\frac{1}{2}\lambda_2 a^2 + \frac{1}{2}\lambda_2 a^2} \right) S$

II. Operations for Contravariant Forms

As there are two different symbols to be employed for a Contravariant, depending upon which Coefficient we decide to call our leading Coefficient, so there are different Symbols in the Case of the Contravariants depending also upon which Coefficient we prefer to select as our leading one.

Let us therefore, agree that henceforth in Contravariants Capital

we that we have found by means of the considerations that it is the

Coefficient of $\frac{\lambda^2}{2}$ in the expansion

(See Elliott's Algebra of Quantics p. 44 &c.)

$$= \left(-\frac{\Xi_1}{\Xi_0} - \frac{\Xi_2}{\Xi_0}\Omega_{ba} - \frac{\Xi_2}{\Xi_0}\Omega_{ca}\right) S.$$

or $\frac{\Xi_1^2}{\Xi_0^2} S - \frac{\Xi_0 \Xi_1}{\Xi_0^2}\Omega_{ba} S - \frac{\Xi_0 \Xi_2}{\Xi_0^2}\Omega_{ca} S + \frac{1}{2}\frac{\Xi_1^2}{\Xi_0^2}\Omega_{ba}^2 S$

$+ \frac{\Xi_1 \Xi_2}{\Xi_0^2}\Omega_{ba}\Omega_{ca} S + \frac{1}{2}\frac{\Xi_2^2}{\Xi_0^2}\Omega_{ca}^2 S.$

We see here that we get the terms in Ξ_1, Ξ_2 by operating with Ω_{ba} and Ω_{ca} as though they were the Symbols of Differentiation with $\frac{1}{2}$ or $\frac{1}{3}$ to get the terms $\Xi_0 \Xi_3$ &c. also $\Omega_{ba}\Omega_{ca}$ &c.

4 Characteristic
Equations Prove that
their Expansions Power-
Invariants a Theorem of
Sign. The Signs being
always positive
The Co-efficients for
Equations-Critical Roots Power
Terms always positive
both pos.ve and negative
Terms
By the above Rules
it be seen a Cubic Equation
or Cube-Quadratic will have
three Real Roots by 4° 3°
2° respectively Real Roots
Quad Equation Real Roots
to two Terms,
For a Quadratic the Expression

Mixed Forms or mixed Concomitant

The process of finding the Concomitants or mixed forms requires more calculation than either the Covariant or Contravariant forms.

As the mixed forms are made up of both Point and Line Coordinates we must naturally substitute two Operators to the Symbolic Letters in each case.

Referring to [illegible] page 411 we see that in the Line Operator to be substituted for the Symbolic Letter

Hard Core Reading Coefficient
Weight Mean Weak $\sum x_i^c \bar{y}_i^+$
Or $x_0^c \bar{y}_0^+$ Core formula by
Expanded Upper Weak Function

§ 7.

Relation between the Alternants

We have shown that various types of identities for Determinants, such as $\overline{12}\cdot\overline{34} + \overline{13}\cdot\overline{42} + \overline{14}\cdot\overline{23} = 0$ etc. is [illegible] to the introduction [illegible] a [illegible] [illegible] [illegible] [illegible]

By identity of the [illegible] [illegible] [illegible]

It is to be observed that the [illegible] is used that is known as Einstein's convention. To say that $1\bar{2}3\bar{4} + \bar{1}3\bar{4}2 + \bar{1}4\bar{2}3$ [illegible] we have to simply write the corresponding [illegible] we have 1, 2, 3, 4 corresponding to our coefficients taken in order.

Take the case of 2 Covariants, Remembering our Coefficients

200 020. 002. 011. 101. 110

1 2 3 4 5 6.

Saying that $= 1\bar{5}4 + 1\bar{3}4\bar{2} + \bar{1}4\bar{2}3$ is equivalent to saying that the determinant $\bar{u}_{200}\bar{u}_{020}\bar{u}_{002}\bar{u}_{011} +$ $\bar{u}_{200}\bar{u}_{020}\bar{u}_{011}\bar{u}_{002} + \bar{u}_{200}\bar{u}_{011}\bar{u}_{020}\bar{u}_{002}$ vanishes.

I. Calculation of the Linear Combinants of 2 Conics

Thus it may have in my disposal rank for use in practice desirable to find the linear Combinants of two Conics. It is similar form as the Eliminant between the Conics "Quantum" in algebra $(a-x)(b-x)$.

Let our two Conics be written in the forms

$a_{200}x_0^2 + a_{020}x_1^2 + a_{002}x_2^2 + 2a_{011}x_1x_2 + 2a_{101}x_0x_2 + 2a_{110}x_0x_1$, and

$b_{200}x_0^2 + b_{020}x_1^2 + b_{002}x_2^2 + 2b_{011}x_1x_2 + 2b_{101}x_0x_2 + 2b_{110}x_0x_1$.

The coefficients of $\det(A - \lambda I)$ are known as the fifth determinants formed from the matrix of the coefficients of the two equations. The determinants that do not vanish are invariant under the substitution of multiples already in front.

Let us call the coefficient of x_0^2 zero by changing coefficients, otherwise the coefficient of x_1^2 or x_2^2 is unchanged.

Since this is a mixed covariant of f_1, f_2 each coefficient will have only two symbolical

two more - after the second requirement that the Electricians have their responsibilities clearly defined.

Operating with $\overline{200.011}$ we get $\overline{200.022}$ which does not vanish. Operating with $\overline{200.020}$ we get $\overline{200.040}$ which also does not vanish. We thus see that $\overline{200.011}$ is not An end Coefficient.

Applying the same test to $\overline{110}\,\overline{101}$ by operating with $\overline{\text{I}}$ we get $\overline{\text{I}}\,\overline{\text{I}}\,\overline{\text{I}}\,

With Ω_{01} we get $\overline{110}\cdot\overline{110}$ which vanishes also.

Our leading coefficient then is $\overline{110}\cdot\overline{101}$.

To find the remaining terms in Ξ_0 we expand the terms:

$$X_0^2\left(e^{\frac{X_2}{X_0}\Omega_{02}+\frac{X_1}{X_0}\Omega_{01}}\right)S.$$

or $X_0^2 + X_0X_1\Omega_{01}S + X_0X_2\Omega_{02}S + \frac{1}{2}\Omega_{01}^2X_1^2S$
$+ X_1X_2\Omega_{01}\Omega_{02}S + \frac{1}{2}\Omega_{02}^2X_2^2S + \dots = \Xi_0 = \Xi_0''$

We see from this expansion that the next term in Ξ_0 is $X_0X_1\Xi_0$. To get this we must operate with Ω_{01} on $\overline{110}\cdot\overline{101}$

$\Omega_{01} = \overline{020}\cdot\overline{101} + \overline{110}\cdot\overline{011}$ whose coefficient is $X_0X_1\Xi_0$

For x_2 in E_1 we operate with L_{x_2} on $|10 \cdot 101|$ getting
$011 \cdot 01 + 110 \cdot 00 = 0$. For coefficient of x_1^2 in E_1 we operate with L_{x_2} twice and divide the result by two getting
$020 \cdot 011$.
For coefficient of $x_1 x_2$ in E_1 operate with L_{x_2} getting
$020 \cdot 012$. For the Lemma to hold operating with $\frac{1}{2} L_{x_2}$ on $|10 \cdot 101|$ must result in the coefficient of x_1^2 in E_1.

Having the terms in E_1 we can get the terms for E_2 and E_3 by alternations. The complete specifications of these steps of the method of bounding will be found

$\overline{110} \cdot \overline{101}$ commutated by $\Omega_{bc} \to \Omega_{ab}$

$\Omega_{ab} = 0\overline{20} \cdot \overline{101} + \overline{110} \cdot \overline{011}$

$\Omega_{ac} = \overline{101} \cdot \overline{101} + \overline{110} \cdot \overline{002}$

$\frac{1}{2}\Omega_{ab}^2 = 0\overline{20} \cdot \overline{011}$

$\Omega_{ab}\Omega_{ac} \quad 0\overline{20} \cdot \overline{002}$

$\frac{1}{2}\Omega_{ac}^2 \quad 0\overline{11} \cdot \overline{002}$

$\overline{101} \cdot \overline{200}$

$0\overline{11} \cdot \overline{200} + \overline{101} \cdot \overline{110}$

$0\overline{02} \cdot \overline{200}$

$0\overline{11} \cdot \overline{110}$

$\overline{110} \cdot \overline{002} + 0\overline{11} \cdot \overline{101}$

$00\overline{2} \cdot \overline{101}$

$\overline{200} \cdot \overline{110}$

$\overline{200} \cdot 0\overline{20}$

$\overline{101} \cdot \overline{110} + \overline{200} \cdot \overline{011}$

$\overline{110} \cdot 0\overline{20}$

$\overline{101} \cdot 0\overline{20} + \overline{110} \cdot 0\overline{11}$

$\overline{200} \cdot \overline{110}$

1. Permutation of the terms.

There are different methods of permutation which may be resorted to. Among the Cyclic permutation is a very convenient one to use.

By a Cyclic permutation Each term is to move in rotation; the coefficients $\bar{1}10\cdot\bar{1}01$ for example going into $\bar{1}11\cdot\bar{1}10\cdot 020\bar{1}11$ by a cyclic permutation becomes $00\bar{2}\cdot 101$

Taking a term such as $x_0^2 x_1$ which may be written as $x_0^2(x_1 - x_{-1})$ and

letting 0 go into 1 once I get 1, a but 0 the other term becomes $x_1^2 x_2 y_0 - x_1^2 = 1$ or $x_1^2 \bar{y}_1$

In the same way $x_0 x_1 y_0$ becomes $x_1 \bar{y}_1$

$x_0 x_2$ " " $x_0 x_1 \bar{y}_1$

x_1^2 " " $x_1 \bar{y}_1$

$x_1 x_2$ " " $x_0 x_2 \bar{y}_1$

x_2^2 " " $x_0^2 \bar{y}_1$

The terms in $\bar{y}$ can be gotten from y in the same way that the terms in y were gotten by the $\bar{y}$ terms by changing from $\bar{y}$ to y, remembering this time that each term is to be subtracted from 1

multiple of four as before.
110.101 by this method of
permutations goes into
$10,011$ to which we shall
call the complement of x_{n-2}
$01,00$ goes into 110.020
which is the complement of
$x_1 x_2$
Thus $x_0 x_1$ permutes x_{n-2}

$x_0 x_1$ " " $x_1 x_{n-2}$
$x_0 x_{n-1}$ " " $x_1 x_{n-1}$
x_2 " " x_3
$x_2 x_{n-1}$ " " $x_3 x_{n-1}$
x_2 " " x_1 .

Other methods of permutation

Other methods of permutation generate sequences which are not lexicographic but have the property that each method usually involves a sequence of steps like the lexicographic ones but the steps are not arranged.

If we write $\frac{3}{30}$ in the form $\frac{3}{3 \cdot 10}$ we can start by dividing 3 into 30 to get 10 and then get $\frac{1}{10}$ which is simpler.

We can now relatively
be able to see in 3_1
from 3_0. That can be written
as the knot interchange
I can.

If we see 3_1 as
$\#\ 3_0$ by interchanging the
first and second
[illegible] and the [illegible]
Similarly, 5_1 3_0 [illegible] with
5_1 3_1. We obtain
[illegible]
$\#\ 3_0$ [illegible] to 020·101 +
110·011 to get the Conjecture
$\#\ 5_1$ 3_1 we interchange
the fourth and [illegible]
[illegible] and [illegible]
Here Our [illegible] is
011 200 - [illegible]

By some Virtue Change of
Colour & round a Change
of Bigness we get Curves in
$\mathbb{Z}_7$
Thus, $k_0^2 \mathbb{Z}_0$ goes into $x_2^2 \mathbb{Z}_2$
$x_0 x_1 \mathbb{Z}_0$ goes into $x_0 x_1 \mathbb{Z}_2$ by
interchanging the Fourth
Simple Shrinks [illegible] Curves
by changing the Bigness of
the Curves

Methods of Localization
Relating the Space
Coordinates

Commutative Property of the Cross Product

If the Cross product of two vectors is equal to the cross product of the same two vectors in the opposite order, then the cross product is commutative.

Quadratic Relations

We say that there is a linear combination of Single linear combinations which we may abbreviate as follows: $|ab_{-}|(x_{-},y_{-}) = \pm g|x_{-}y_{-}|^2$. The coefficients of the combinations are known as the fifteen two-rowed determinants $ik = \begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix}$.

But $|x_{-}y_{-}|^2$ has eight terms and the coefficients. There must be some relation between the coefficients.

These linear relations are given by the vanishing of $(x-1)(x)$ the only invariant $= (x-3)(x)^2$.

This covariant can be found by finding the leading coefficient after making the above substitutions and the coefficients of the covariant.

We get for our leading coefficient

$(a_0 + a_1 x_1 = 2x_0 x_1)$.

The coefficient of x_0 is $a_0 a_0^2 + 3 a_0 a_1 - 3 a_2 =$ which is the resolvable expression of the Cubic itself.

$+ x_0^2 \; 3 a_0 + x_0 x_1 \; 3 a_1 + x_1 x_2 \; 3 a_2$.

Substituting the coefficients of these terms in our leading coefficient and taking binomial coefficients into account we get:

$$\left[110 \cdot 101 + 011 \cdot \underline{200} + 101 \cdot \underline{110} + \underline{101 \cdot 110} + \underline{200 \cdot 011}\right] x,$$

which vanishes.

The conclusion, then, that $(a + \frac{1}{2}(1-x))^3 x$ can be written in the form $(1 - \frac{1}{2}(1-x))^2 \gamma$ is that the linear combination of which $a_2(x)$ denotes the

Further, the following
determinants are subject
to similar quadratic
relations of the form
$12\cdot\overline{34} + 13\cdot\overline{42} + 14\cdot\overline{23} = 0$
and any similar
determinants subject
to these similar
relations are proportional
to equal zero
determinants.
The only quadratic
invariants of
$|a_{ij}|(x)^2 - |a_{ij}|(y)^2$ are

Three Cases

Calculation of Constants

Let us now consider the linear Combinations of three Cases.

Since our Cases have sixteens, And we select three at a time The number of determinants from which we must Select our leading Coefficients is [illegible] Equality.

We find that $\Delta = 2\Delta\Delta \cdot 110, 111$ has four Annihilators.

$\Sigma_1 \Sigma_2 \Sigma_3 \Sigma_4 \Sigma_{24}$

We see that this coefficient has the weight 411. This is the weight of x_0^3 when the latter is multiplied by $(\frac{3}{0} x)$. Thus we get $200.110.020$ as the coefficients of x_0^3.

In order to get the remaining terms we expand

$$\overset{3}{\underset{0}{X}} \cdot e^{\left(\frac{x_2}{x_0}\Omega_{ac} + \frac{x_1}{x_0}\Omega_{ab}\right)} \cdot$$

$$0 \times x_0^3 3 + x_0^2 x_1 \Omega_{ac} 0 + x_0 x_1 \Omega_{ab} 3 + \tfrac{1}{2} x_0 x_1^2 \Omega_{ab}^2$$
$$+ x_0 x_2 \Omega_{ac} \Omega_{ab} 3 + \tfrac{1}{2} x_0 x_1^2 \Omega_{ac}^2 0 + \tfrac{1}{6} x_1^3 \Omega_{ab}^3$$
$$+ \tfrac{1}{2} x_0 x_2^2 \Omega_{ac}^2 \Omega_{ab} 3 + \tfrac{1}{2} x_0 x_1 x_2 \Omega_{ac} \Omega_{ab}^2 1 + \tfrac{1}{6} x_1^3 \Omega_{ab}^3 0$$

The combinants is given in full on the following page.

$\bar{2}\bar{0}\bar{0}\cdot\bar{1}\bar{1}\bar{0}\cdot\bar{1}\bar{0}\bar{1}$ annihilates by $\Omega_{ca}\Omega_{ab}\Omega_{ca}\Omega_{ab}$ | x_0^2

$\Omega_{ab} = \bar{2}\bar{0}\bar{0}\cdot 0\bar{2}0\cdot\bar{1}0\bar{1} + \bar{2}\bar{0}\bar{0}\cdot\bar{1}\bar{1}0\cdot 0\bar{1}\bar{1}$ | $x_0^2 x_1$

$\frac{1}{2}\Omega_{ab}^2 = \bar{1}\bar{1}0\cdot 0\bar{2}0\cdot\bar{1}0\bar{1} + \bar{2}\bar{0}\bar{0}\cdot 0\bar{2}0\cdot 0\bar{1}\bar{1}$ | $x_0 x_1^2$

$\frac{1}{3}\Omega_{ab}^3 = \bar{1}\bar{1}0\cdot 0\bar{2}0\cdot 0\bar{1}\bar{1}$ | x_1^3

$-\Omega_{ac} = \bar{2}\bar{0}\bar{0}\cdot 0\bar{1}\bar{1}\cdot\bar{1}0\bar{1} + \bar{2}\bar{0}\bar{0}\cdot\bar{1}\bar{1}0\cdot 00\bar{2}$ | $x_0^2 x_2$

$\frac{1}{2}\Omega_{ac}^2 = \bar{2}\bar{0}\bar{0}\cdot 0\bar{1}\bar{1}\cdot 00\bar{2} + \bar{1}0\bar{1}\cdot\bar{1}\bar{1}0\cdot 00\bar{2}$ | $x_0 x_2^2$

$\frac{1}{3}\Omega_{ac}^3 = \bar{1}0\bar{1}\cdot 0\bar{1}\bar{1}\cdot 00\bar{2}$ | x_2^3

$\Omega_{ab}\Omega_{ac} = 2|\bar{1}\bar{1}0\cdot 0\bar{1}\bar{1}\cdot\bar{1}0\bar{1}| + \bar{2}\bar{0}\bar{0}\cdot 0\bar{2}0\cdot 00\bar{2}$ | $x_0^2 x_1 x_2$

$\frac{1}{2}\Omega_{bc}^2\Omega_{ab} = \bar{1}\bar{1}0\cdot 0\bar{1}\bar{1}\cdot 00\bar{2} + \bar{1}0\bar{1}\cdot 0\bar{2}0\cdot 00\bar{2}$ | $x_1^2 x_2$

$x_1 x_2^2$

Since we had twenty determinants and there are only ten terms in this, we should expect another with ten terms which could be found on the next page. We find that $2\bar{0}\bar{0}\cdot\bar{1}\bar{1}0\cdot 0\bar{2}\bar{2}$ is annihilated by $\Omega_{bc}\Omega_{ab}\Omega_{bc}\Omega_{ab}$ It is therefore the coefficient of x_0^2

$\overline{200}\cdot\overline{110}\cdot\overline{020}$	u_2^3
$-\Omega_{bc} = 2\lvert\overline{110}\cdot\overline{200}\ \overline{011}\rvert + \overline{101}\ \overline{200}\ \overline{020}$	$u_2^2 u_1$
$+\frac{1}{2}\Omega_{bc}^2 = 2\lvert\overline{200}\ \overline{101}\ \overline{011}\rvert + \overline{200}\ \overline{110}\ \overline{002}$	$u_2 u_1^2$
$+\Omega_{ac}\Omega_{bc} = 4\lvert\overline{101}\ \overline{110}\ \overline{011}\rvert + \overline{200}\ \overline{002}\ \overline{020}$	$u_0 u_1 u_2$
$-\Omega_{ac} = 2\lvert\overline{110}\ \overline{101}\cdot\overline{020}\rvert + \overline{011}\ \overline{200}\ \overline{020}$	$u_2^2 u_0$
$\frac{1}{2}\Omega_{ac}^2 = 2\lvert\overline{101}\ \overline{011}\ \overline{020}\rvert + \overline{002}\ \overline{110}\ \overline{020}$	$u_0^2 u_2$
$-\frac{1}{2}\Omega_{ac}^2\Omega_{bc} = 2\lvert\overline{110}\ \overline{101}\ \overline{002}\rvert + \overline{002}\ \overline{200}\ \overline{011}$	$u_1^2 u_0$
$-\frac{1}{2}\Omega_{ac}^2\Omega_{bc} = 2\lvert\overline{110}\ \overline{002}\ \overline{011}\rvert + \overline{002}\ \overline{101}\ \overline{020}$	$u_0^2 u_1$
$-\frac{1}{6}\Omega_{bc}^3 = \overline{101}\ \overline{200}\ \overline{002}$	u_1^3
$-\frac{1}{6}\Omega_{ac}^3 = \overline{011}\ \overline{002}\cdot\overline{020}$	u_0^3

Since our Curves are 2 dimensional we have 2 independent parameters which is sufficient for our purpose. The first of above is the locus of double points of the net or Hessian. The second is the Cayley curve.

* The operators to be used in the above Covariant were got by the Expansion of $\frac{1}{2}\{e^{-\frac{u_0}{u_2}\Omega_{ac} - \frac{u_1}{u_2}\Omega_{bc}}\}$

* See Hunt p. 100.

Quadratic Values

The elements are subject to certain quadratic relations of the form
$123.456 - 124.356 + 134.256$
$- 234.156 = 0$; however,
there are only five
three independent which
will be shown later.
There are also identities
of the form $512.034 + 513.045 -$
$+ 514.523 = 0$. We notice that
that we lack one figure,
so 5, first and permute
the remaining four. We
can select this four in 6 ways
and this we can repeat
one in four ways.

[illegible] to [illegible]
[illegible] light
[illegible] we have 30
Conditions to [illegible]
[illegible] must [illegible] be
a [illegible], [illegible] the
[illegible] of the [illegible]
[illegible] [illegible] [illegible]
[illegible] [illegible]

Writing the Statement and
the Conclusion in the forms
$x \in J$ and (-5) respectively, and
separating the [illegible]
of Each, and the [illegible]
pieces [illegible] of
[illegible] [illegible] [illegible]
following list:

$(\alpha x)^3 \cdot (\beta x)^3 \cdot$

$(a_3)^3 (b_3)^3 \cdot$

$(\alpha x)^3 \cdot (a_3)^3 \cdot$

$\alpha\beta_3|^2 (\alpha x)(\beta x)$

$|ab_3|^2 (a_3)(b_3)$

$(a\alpha)(\alpha x)^2 (a_3)^2 \cdot$

$(a\alpha)^2 (\alpha x)(a_3) \cdot$

$(a\alpha)^3 \cdot$

The three conics may be considered as the polar conics of a cubic.

Let us write our cubics in the form

$a(x_0^3 + x_1^3 + x_2^3) + 6mx_0x_1x_2$

Forming the polar conics of the pts $1,0,0;\ 0,1,0,\ 0,0,1$ respectively our conics become (1) $ax_0^2 + 2mx_1x_2$

(2) $ax^2 + 2mkx_0x_1$ (3) $ax_0^2 + 2mx_0x_1$

The first three invariants are sextics, and the only Salmon twenty eight bitangents are thirty four conditions required, so we may eliminate them from present consideration.

Observe here the new notations of our conics due to the fact that we have considered them as the polar conics of the cubic $a(x_0^3 + x_1^3 + x_2^3) + 6mx_0x_1x_2$

Our conics under this new notation are

(1) $2x_0^2 + 2mx_1x_2$

(2) $ax_1^2 + 2 \ldots x_1 x_2$ (3) $-x_2^2 + 2mx_2x_3$

Four invariants with. We shall consider we in the old notation

(1) $|\alpha \beta \gamma|^2 \alpha x \beta(\beta x)$.

(2) $|\ldots x|^2 (a_3)^{-1/3}$.

(3). $(a\alpha)(\alpha x)^2 (a_3)^{-1}$.

(4) $(a\alpha)^2 (\alpha x)(a_3)$.

(5). $(a\alpha)^3$.

Under the new notation they become:

1. $-\frac{(2m^3 + a^3)}{18}$.

(2) $-\frac{(4m^3 - a^3)}{18}$.

(3) $m^2 a^3$

4. $\frac{(2m^3 + a^3)(4m^3 - a^3)}{18} - m^2 a^3$

5). $3m^2 a^3 + \frac{(2m^3 + a^3)(4m^3 - a^3)}{6}$

Taking λ_1 times the first
inequality, λ_2 times the
second, and so forth
we get

$$\lambda_1[-2m^3+a^3]^2 + \lambda_2[-4m^3-a^3]^2 + \lambda_3[\quad\quad]^2$$
$$+\lambda_4[1+m^2a^3+(3m^3+a^3)(4m^3-a^3)]$$
$$+\lambda_5[54m^3a^3+36(2m^3+a^3)(4m^3-a^3)] \geq 0$$

$$\therefore (-4\lambda_1 - 16\lambda_2 + 8\lambda_4 + 24\lambda_5)m^6$$
$$+(-2\lambda_1+4\lambda_2+9\lambda_3+10\lambda_4+30\lambda_5)m^3a^3$$
$$+(\lambda_1+\lambda_2+\lambda_4+3\lambda_5)a^6 \geq 0$$

Equating to zero the
coefficients of m^6, m^3a^3 and
a^6 [illegible] the following
three equations:

The 2 Syzygies, Theorem
(1) meeting the Covariants
& its Combinants are:

(1) $3(a\alpha)^2(\alpha x)a_3 - (a\alpha)^3(ax) = 0$

and

(2) $-2\left|\begin{matrix} \alpha & \beta \\ a & b \end{matrix}\right|^2(\alpha x)(\beta x) + \left|\begin{matrix} \alpha & \gamma \\ a & b \end{matrix}\right|^2(a\beta)^2 a_\gamma$
$-2(a\alpha)(\alpha x)^2(a\beta)^2 + (a\alpha)^2(\alpha x)(a\beta)a_\beta = 0.$

These two forms are
normal. To say that a
form is normal implies
that if we operate upon
it with (βx) the Transvector
The two Syzygy is a
(1.1) form and vanishes
identically because of these
Conditions, but I am

it is a Normal Form
Duty Satisfies & Conditions.

The Second Step is —
(2,2) Form a List of Work Ordinary
Satisfy Bo Conditions.
Here we of make on the list
(3 X) it is annex to a
1;1) form with Variables.
It is, therefore, Normal
Also, and Since the (1.1)
form Connects of 9 Lines it
Only Satisfies 36-9, or 27,
Conditions
But Do Not Have to Have
Satisfy then 27 +
35- Conditions which was
The Number - Required

The fifteen identities of the

$$123.456 - 124.356 + 134.256 - 234.156 = 0$$

are the following:

1. $\overline{345}\,\overline{612} - \overline{346}\,\overline{512} + \overline{356}\,\overline{412} - \overline{456}\,\overline{312} = 0 = a$
2. $\overline{245}\,\overline{613} - \overline{246}\,\overline{513} + \overline{256}\,\overline{413} - \overline{456}\,\overline{213} = 0 = b$
3. $\overline{235}\,\overline{614} - \overline{236}\,\overline{514} + \overline{256}\,\overline{314} - \overline{356}\,\overline{214} = 0 = c$
4. $\overline{234}\,\overline{615} - \overline{236}\,\overline{415} + \overline{246}\,\overline{315} - \overline{346}\,\overline{215} = 0 = d$
5. $\overline{234}\,\overline{516} - \overline{235}\,\overline{416} + \overline{245}\,\overline{316} - \overline{345}\,\overline{216} = 0 = e$
6. $\overline{145}\,\overline{623} - \overline{146}\,\overline{523} + \overline{156}\,\overline{423} - \overline{456}\,\overline{123} = 0 = f$
7. $\overline{135}\,\overline{624} - \overline{136}\,\overline{524} + \overline{156}\,\overline{324} - \overline{356}\,\overline{124} = 0 = g$
8. $\overline{134}\,\overline{625} - \overline{136}\,\overline{425} + \overline{146}\,\overline{325} - \overline{346}\,\overline{125} = 0 = h$
9. $\overline{134}\,\overline{526} - \overline{135}\,\overline{426} + \overline{145}\,\overline{326} - \overline{345}\,\overline{126} = 0 = i$
10. $\overline{123}\,\overline{645} - \overline{126}\,\overline{345} + \overline{136}\,\overline{245} - \overline{236}\,\overline{145} = 0 = j$
11. $\overline{125}\,\overline{634} - \overline{126}\,\overline{534} + \overline{156}\,\overline{234} - \overline{256}\,\overline{134} = 0 = k$
12. $\overline{124}\,\overline{635} - \overline{126}\,\overline{435} + \overline{146}\,\overline{235} - \overline{246}\,\overline{135} = 0 = l$
13. $\overline{124}\,\overline{536} - \overline{125}\,\overline{436} + \overline{145}\,\overline{236} - \overline{245}\,\overline{136} = 0 = m$
14. $\overline{123}\,\overline{546} - \overline{125}\,\overline{346} + \overline{135}\,\overline{246} - \overline{235}\,\overline{146} = 0 = n$
15. $\overline{123}\,\overline{456} - \overline{124}\,\overline{356} + \overline{134}\,\overline{256} - \overline{234}\,\overline{156} = 0 = o$

That the first four are independent may be seen in the following way. Considering them as members of the identity consisting of ten terms

$: 1\cdot\overline{23}\cdot\overline{45}\overline{6} + 2\overline{34}\cdot\overline{15}\overline{6} + 1\overline{34}\cdot\overline{25}\overline{6} + 1\overline{24}\cdot\overline{35}\overline{6}$
$+ 1\overline{25}\cdot\overline{34}\overline{6} + 12\overline{6}\cdot\overline{35}\overline{4} + 1\overline{5}\cdot\overline{26}\overline{34} + 1\overline{4}\cdot\overline{26}\overline{35}$
$+ 1\overline{36}\cdot\overline{245} + 1\overline{46}\cdot\overline{253} = 0$ we get the matrix

-1	0	0	-1	-1	-1	0	0	0	0
1	0	1	0	0	0	1	0	1	0
0	0	-1	-1	0	0	0	-1	0	-1
0	1	0	0	1	0	1	1	0	0
0	1	0	0	0	-1	0	0	-1	-1

Since the first 5 rowed determinant reduces to -2, the matrix is not 0 so the four 5-Eq relations are independent.

We shall now replace the last term in each of the first five. Carrying out the substitutions there will be many cancellations, and putting our unknowns α, β, γ or λ, μ, ν we get from the first five:

(1) $\overline{23}\cdot\overline{45} + \overline{13}\cdot\overline{46} + \overline{14}\cdot\overline{36} = \overline{34}\cdot\overline{61} - \alpha$

(2) $\overline{14}\cdot\overline{26} + \overline{13}\cdot\overline{46} = \overline{46}\cdot\overline{12} - \overline{46}\cdot\overline{12} + \beta$

(3) $-\overline{13}\cdot\overline{26} - \overline{14}\cdot\overline{36} = \overline{36}\cdot\overline{14} - \overline{36}\cdot\overline{14} + \gamma$

(4) $-\overline{34}\cdot\overline{16} + \overline{46}\cdot\overline{12} = \overline{36}\cdot\overline{41} - \overline{46}\cdot\overline{31} + \lambda$

(5) $\overline{34}\cdot\overline{16} = \overline{35}\cdot\overline{41} - \overline{45}\cdot\overline{31} + \overline{46}\cdot\overline{21} + \nu$

$\therefore \alpha + \beta + \gamma = \lambda - \nu$

$\delta + \alpha = \beta - \lambda + \mu$

$-\beta - \gamma = \beta - \lambda + \nu$

$-\epsilon + \lambda = -\beta + \gamma + \mu$

$\epsilon = -\lambda + \nu - \mu + \nu$

$\therefore$

$\therefore \alpha = p - r + s - \frac{1}{2}(a - b - c + d + e)$

$\beta = -p + q + r - s - t + d + e.$

$\gamma = p - q + t - \frac{1}{2}(a + b + c + d + e).$

$\delta = -p + q - s + \frac{1}{2}(a + b - c + d + e).$

$\epsilon = -p + r - t + e$

Substituting these values in the identities:

$f = \frac{1}{2}(a - b - c + d + e) = 0$

$g = -\frac{1}{2}(a + b + c + d + e) = 0$

$h = \frac{1}{2}(a + b - c - d - e) = 0$

$i = -\frac{1}{2}(a + b - c + d + e) = 0$

$j = -\frac{1}{2}(a - b - c + d + e) = 0$

$k = -\frac{1}{2}(a + b - c - d - e) = 0$

$l = \frac{1}{2}(a + b + c + d + e) = 0$

$m = -\frac{1}{2}(a + b + c - d - e) = 0$

$n = +\frac{1}{2}(a - b - c - d - e) = 0$

$o = -\frac{1}{2}(a - b + c + d - e) = 0.$

Combinant = 2 + Covariant

I. Calculation of Combinant.

The theory of 4 covers is very closely related to the theory of two. Having our leading coefficient we can use Elliott's operator and get the remaining coefficients or we may derive our coefficients from the combinant of 2 covers.

By experiment we find the leading coefficients of $x_0 u_0^4$ to be

$$120\bar{1}0.0\bar{2}0.0\bar{2}2.0\bar{1}11 \ldots$$

* Elliott's operator in our notation is $x_0\left(e^{\frac{x_1}{x_0}\left(\frac{}{ab}\right) + \frac{x_2}{x_0}\left(\frac{}{ac}\right)}\right) = x_0 J + x_1 \frac{}{ab} J + x_2 \frac{}{ac} J$.

Operating with $x_0 J + x_1 \frac{}{ab} J + x_2 \frac{}{ac} J$ on $120\bar{1}0.0\bar{2}0.0\bar{2}2.0\bar{1}11$

we get $x_0\left(2\bar{0}00.0\bar{2}0.00\bar{2}.0\bar{1}11\right) + 2x_1\left(1\bar{1}00.0\bar{1}0.00\bar{1}.0111\right)$

$+ 2x_2\left(10\bar{1}.0\bar{2}0.0\bar{2}.0\bar{1}11\right)$

$$u_0^4\left\{e^{-\frac{u_2}{u_0}\left(\frac{}{ab}\right) - \frac{u_1}{u_0}\left(\frac{}{ac} - \frac{1}{2}\frac{}{a}\left(\frac{}{ab}\right)\right)}\right\} \frac{}{dx_0} e^{\frac{x_1}{x_0}\frac{}{ab} J + \frac{x_2}{x_0}\frac{}{ac} J} J$$

Now observing that this is the $e^{-\frac{u_2}{u_0}(k_{20} - x_2 \frac{k_2}{x_{20}}) - \frac{u_1}{u_0}(k_{10} - x_1 \frac{k_1}{x_{10}})}$ we get the term in $u_1 u_2$.

$$x_0 \{2|200.020 \cdot 10\bar{1} \cdot 0\bar{1}11 + 20\bar{2}.02\bar{0}.00\bar{2} \cdot 110\}$$
$$+ 2x_1 \{2\bar{1}10.02\bar{0} \cdot 10\bar{1} \cdot 0\bar{1}11\} + x_2 \{1\bar{2}00.0\bar{2}0.00\bar{2} \cdot 011\}$$
$$+ 2|10\bar{1}\ 0\bar{2}0\ 00\bar{2} \cdot 1\bar{1}01\},$$

Having these terms we can, by permutation get the entire Combinant.

We note here that it is of the fourth order and Weight while that of two Circles is of the second order, and first class.

The entire Combinant is given on the following page.

x_0

$2\bar{0}0\ 0\bar{2}0\ 00\bar{2}\ 0\bar{1}1$ annihilated by $u_0^2 u_2$ u_0^2

$2|00\bar{2}\cdot 2\bar{0}0\cdot 0\bar{1}1\cdot 1\bar{1}0|1 - 00\bar{2}\,2\bar{0}0\,0\bar{2}0\,\bar{1}01$ $u_0 u_1$

$2|0\bar{2}0\cdot 0\bar{1}1\ 2\bar{0}0\ 1\bar{0}1|1 - 0\bar{2}0\ 00\bar{2}\ 2\bar{0}0\ 1\bar{1}0$ $u_0 x_2$

$2|1\bar{1}0\ 0\bar{0}2\ 2\bar{0}0\ 1\bar{0}1|1$ u_1^2

$4|2\bar{0}0\ 1\bar{1}0\ 1\bar{0}1\cdot 0\bar{1}1|1$ $u_1 u_2$

$1\bar{0}1\ 2\bar{0}0\ 0\bar{2}0\cdot 1\bar{1}0$ u_2^2

x_1

$2|1\bar{1}0\cdot 0\bar{2}0\ 0\bar{0}2\ 0\bar{1}1|1$ u_0^2

$2|00\bar{2}\ 1\bar{0}1\ 0\bar{2}0\ 1\bar{1}0|1 - 00\bar{2}\,2\bar{0}0\,0\bar{2}0\,\bar{1}11$ $u_0 u_1$

$4|0\bar{2}0\cdot 0\bar{1}1\ 1\bar{1}0\ 1\bar{0}1|1$ $u_0 u_2$

$0\bar{2}0\cdot 00\bar{2}\ 2\bar{0}0\ 1\bar{0}1$ u_1^2

$2|2\bar{0}0\cdot 0\bar{2}0\cdot 1\bar{0}1\ 0\bar{1}1|1 - 2\bar{0}0\ 0\bar{2}0\ 00\bar{2}\ 1\bar{1}0$ $u_1 u_2$

$2|0\bar{1}1\ 2\bar{0}0\ 0\bar{2}0\ 1\bar{1}0|1$ u_2^2

x_2

$2|1\bar{0}1\cdot 0\bar{2}0\cdot 00\bar{2}\cdot 0\bar{1}1|1$ u_0^2

$4|00\bar{2}\cdot 1\bar{0}1\cdot 0\bar{1}1\cdot 1\bar{1}0|1$ $u_0 u_1$

$2|0\bar{2}0\cdot 00\bar{2}\ 1\bar{1}0\ 1\bar{0}1|1 - 0\bar{2}0\ 00\bar{2}\ 2\bar{0}0\ 0\bar{1}1$ $u_0 u_2$

$2|0\bar{1}1\cdot 00\bar{2}\ 2\bar{0}0\ 1\bar{0}1|1$ u_1^2

$2|2\bar{0}0\ 1\bar{1}0\ 00\bar{2}\cdot 0\bar{1}1|1 - 2\bar{0}0\ 0\bar{2}0\ 00\bar{2}\ 1\bar{0}1$ $u_1 u_2$

$00\bar{2}\ 2\bar{0}0\ 0\bar{2}0\ 1\bar{1}0$ u_2^2

As formerly the way to make pass over to the Combinants of 4 Conics from that of two.
Setting the Coefficients of the Conic in order. We give them Signs + − + − − −. If we find that the coefficient of the Corresponding term is made up of 2 or three or 2 negative terms in Order the determinant for the combinant of four Conics the remaining terms taken in order.
If Combinant of 2 conics is made up of positive & negative Terms in order the determinant will be the negative of the remaining Terms. If the terms are positive and negative and rearranged the determinant is positive.

$$\overset{+}{2}\,0\,\overset{-}{5}\,\overset{+}{2}\,0\,\overset{-}{5}\,\overset{+}{1}\,\overset{-}{1}\,\overset{+}{1}\,\overset{-}{1}$$

Coefficient of $x_0^2 x_3^2$ in $I_{01}\,I_{01}$: Coefficient of $x_0^2 x_3^2$ in $2\,0\,5\,2\,0\,5\,1\,1\,1\,1$

II

Geometrical meaning of the Combinant of 4 Conics

Our Combinant is a Covariant of 4 binary quadratic forms

Given 4 Conics as we have there is one & only pencil of Conics touching four lines.

Given a point in a plane there are two Conics of the above pencil which pass through this point.

Polar tangents to these Conics will be the lines corresponding to the point. If one of the lines is given there is only one line and only one Conic of

[illegible] polar vector [illegible]
[illegible] to the [illegible] of the [illegible]
[illegible] $\rho = 2a \cos(\theta - \alpha)$.

We may illustrate with
Steiner's theorem by the
following. Given any 20 inch circle
whose center is at the origin
of coordinates. Inscribe in
it a triangle with fixed
double points at [illegible]
by the [illegible] of the [illegible]
of reduction [illegible]
vertex. [illegible]
be two [illegible] of the
remaining [illegible]
[illegible]
on the [illegible]

[illegible] [illegible] [illegible]
[illegible] [illegible] [illegible]
[illegible] [illegible] [illegible]
[illegible] [illegible] [illegible]
[illegible] [illegible] [illegible]
[illegible] [illegible] [illegible]

Covariants of Forms

1. Calculation of the Covariants.

As we have already seen two cubics $a_{300}x_0^3 + a_{030}x_1^3 + a_{003}x_2^3 + 3a_{210}x_0^2x_1 -$ and $b_{300}x_0^3 + \cdots$ have 2 linear covariants $|ux|^2(vx)^2$ and $|uvw|^3$.

We find that $|\Omega_{10}\Omega_{01}|$ appears in a sigma with $u_0^4u_0$. Besides it has the annihilators Ω_{ve} and Ω_{ev}. It is therefore the leading coefficient.

The expansion of $\Omega_{10}^4\Omega_{01}^4(\frac{1}{2}\frac{1}{x_1} + \frac{1}{x}\frac{1}{x_2})^4$ is

$$x^4 + x^3z\Omega_{xz}\delta + x^3y\Omega_{xy}\delta + \tfrac{1}{2}x^2z^2\Omega_{xz}^2\delta + x^2yz\Omega_{xy}\Omega_{xz}\delta$$
$$+ \tfrac{1}{2}x^2y^2\Omega_{xy}^2\delta + \tfrac{1}{6}xz^3\Omega_{xz}^3\delta + \tfrac{1}{2}xy^2z\Omega_{xz}\Omega_{xy}^2\delta + \tfrac{1}{2}xyz^2\Omega_{xz}^2\Omega_{xy}\delta$$
$$+ \tfrac{1}{6}xy^3\Omega_{xy}^3 + \cdots$$

Operating with operators we obtain above upon the leading coefficient we get all the terms in u_0. The terms in u_1 and u_2 are gotten as in the previous case of the Conics.

Coefficients of $\Omega_{ab}\Omega_{ac}$ in $(a+b)^n(a+c)^n$

$\bar{2}\bar{1}0 \cdot \bar{2}01$ | X_0^4

$\Omega_{ab} = 2[\bar{1}20 \cdot \bar{2}01 + \bar{2}10 \cdot \bar{1}11]$ | $X_0^3 X_1$

$\frac{1}{2}\Omega_{ab}^2 = [\bar{0}30 \cdot \bar{2}01 + 4|\bar{1}20 \cdot \bar{1}11| + \bar{2}10 \cdot \bar{0}21]$ | $X_0^2 X_1^2$

$\frac{1}{3}\Omega_{ab}^3 = 2[\bar{0}30 \cdot \bar{1}11 + \bar{1}20 \cdot \bar{0}21]$ | $X_0 X_1^3$

$\frac{1}{4}\Omega_{ab}^4 = \bar{0}30 \cdot \bar{0}21$ | X_1^4

$\Omega_{ac} = 2[\bar{1}11 \cdot \bar{2}01 + \bar{2}10 \cdot \bar{1}02]$ | $X_0^3 X_2$

$\frac{1}{2}\Omega_{ac}^2 = \bar{0}\bar{1}2 \cdot \bar{2}01 + 4|\bar{1}11 \cdot \bar{1}02| + \bar{2}10 \cdot \bar{0}\bar{0}3$ | $X_0^2 X_2^2$

$\frac{1}{3}\Omega_{ac}^3 = 2[\bar{0}\bar{1}2 \cdot \bar{1}02 + \bar{1}11 \cdot \bar{0}\bar{0}3]$ | $X_0 X_2^3$

$\frac{1}{4}\Omega_{ac}^4 = \bar{0}\bar{1}2 \cdot \bar{0}\bar{0}3$ | X_2^4

$\Omega_{ab}\Omega_{ac} = 2[\bar{0}\bar{2}1 \cdot \bar{2}01 + 2|\bar{1}20 \cdot \bar{1}02| + \bar{2}10 \cdot \bar{0}\bar{1}2]$ | $X_0^2 X_1 X_2$

$\frac{1}{2}\Omega_{ab}^2\Omega_{ac} = 2[\bar{0}\bar{2}1 \cdot \bar{1}11 + \bar{0}\bar{3}0 \cdot \bar{1}02 + 2|\bar{1}20 \cdot \bar{1}02|]$ | $X_0 X_1^2 X_2$

$\frac{1}{6}\Omega_{ab}^3\Omega_{ac} = 2[\bar{0}\bar{3}0 \cdot \bar{0}\bar{1}2]$ | $X_1^3 X_2$

$\frac{1}{2}\Omega_{ab}^2\Omega_{ac}^2 = 2[\bar{1}11 \cdot \bar{0}\bar{1}2 + \bar{1}20 \cdot \bar{0}\bar{0}3 + 2|\bar{0}\bar{2}1 \cdot \bar{1}02|]$ | $X_0 X_1 X_2^2$

$\frac{1}{6}\Omega_{ab}\Omega_{ac}^3 = 2[\bar{0}\bar{2}1 \cdot \bar{0}\bar{0}3]$ | $X_1 X_2^3$

$\frac{1}{4}\Omega_{ab}^2\Omega_{ac}^2 = 3|\bar{0}\bar{2}1 \cdot \bar{0}\bar{1}2| + \bar{0}\bar{3}0 \cdot \bar{0}\bar{0}3$ | $X_1^2 X_2^2$

$\bar{2}\bar{0}1 \quad \bar{3}\bar{0}0$ | x_0^4

$2[\bar{1}\bar{1}1 \cdot \bar{3}\bar{0}0 + \bar{2}\bar{0}1\,\bar{2}\bar{1}0]$ | $x_0^3 x_1$

$0\bar{2}1 \cdot \bar{3}\bar{0}0 + \bar{2}\bar{0}1 \cdot \bar{1}\bar{2}0 + 4|111 \cdot 210|$ | $x_0^2 x_1^2$

$2[\bar{0}21 \cdot \bar{2}10 + \bar{1}\bar{1}1 \cdot \bar{1}\bar{2}0]$ | $x_0 x_1^3$

$0\bar{2}1 \cdot \bar{1}\bar{2}0$ | x_1^4

$2|102\ 3001$ | $x_0^3 x_2$

$00\bar{3} \cdot \bar{3}\bar{0}0 + 3|102\,101|$ | $x_0^2 x_2^2$

$2|003 \cdot 2011$ | $x_0 x_2^3$

$00\bar{3} \cdot 1\bar{0}2$ | x_2^4

$2[\bar{1}\bar{1}1 \cdot 2\bar{0}1 + 0\bar{1}2\ \bar{3}0\bar{0}] + 4[\bar{1}0\bar{2} \cdot 2\bar{1}0]$ | $x_0^2 x_1 x_2$

$4[0\bar{1}2 \cdot 2\bar{1}0] + 2[\bar{1}0\bar{2} \cdot 2\bar{0}1 + \bar{1}0\bar{2} \cdot \bar{1}2\bar{0}]$ | $x_0 x_1^2 x_2$

$2[0\bar{1}\bar{2} \cdot \bar{1}2\bar{0} + 0\bar{2}1\ \bar{1}\bar{1}1]$ | $x_1^3 x_2$

$4[0\bar{1}2\ 2\bar{0}1] + 2[\bar{1}0\bar{2} \cdot \bar{1}\bar{1}1 + 00\bar{3} \cdot \bar{2}\bar{1}0]$ | $x_0 x_1 x_2^2$

$2[00\bar{3} \cdot \bar{1}\bar{1}1 + 0\bar{1}2 \cdot \bar{1}0\bar{2}]$ | $x_1 x_2^3$

$0\bar{0}3 \cdot \bar{1}2\bar{0} + 0\bar{2}1 \cdot \bar{1}0\bar{2} + 4|012 \cdot 1111$ | $x_1^2 x_2^2$

$2|\overline{300}\cdot|\overline{120}|$

$\overline{300}\cdot\overline{030} + 3|\overline{210}\cdot|\overline{120}|$

$2|\overline{210}\cdot\overline{030}| +$

$2[\overline{300}\,|\overline{111}| + \overline{201}\cdot\overline{210}]$

$|\overline{102}\cdot\overline{210} + \overline{300}\cdot\overline{012} + 4|\overline{200}\cdot|\overline{111}|$

$2[\overline{102}\cdot|\overline{111} + \overline{201}\cdot\overline{012}]$

$|\overline{102}\cdot\overline{012}$

$4|\overline{201}\cdot|\overline{120}| + 2[\overline{210}\cdot|\overline{111} + \overline{300}\cdot\overline{021}]$

$2[|\overline{111}\cdot|\overline{120} + \overline{201}\cdot\overline{030}] + 4|\overline{210}\cdot\overline{021}|$

$2[|\overline{111}\cdot\overline{030} + |\overline{120}\cdot\overline{021}]$

$4|\overline{201}\cdot\overline{021}| + 2[\overline{102}\cdot|\overline{120} + \overline{210}\cdot\overline{012}]$

$2[\overline{102}\cdot\overline{021} + |\overline{111}\cdot\overline{012}]$

$|\overline{102}\cdot\overline{030} + |\overline{120}\cdot\overline{012} + 4|\overline{111}\cdot\overline{021}|$

Geometrical meaning of the Covariant

Combinants of two Cubics

In addition to the Combinant already found two Cubics have the Combinant $\kappa b a^3$ This may be shew to be a combinant from the fact that if we write $\lambda b + \mu a$ in place of b in the determinant we get $\lambda^3 \Delta(a b a^3 + \mu a a a^3$ giving us the same result no matter what values may be assigned to λ and μ

Expanding $\int^3 e^{-\frac{\xi}{3}\Omega_{xx} - \frac{\eta}{3}\Omega_{yz}} S^1$

$$+ \int^3 S - \frac{\xi}{3}\int^2 \Omega_{xz} S - \eta \int^2 \Omega_{yz} S + \frac{1}{2}\frac{\xi^2}{3}\int \Omega_{xz}^2 + \frac{2}{3}\eta \int \Omega_{xz}\Omega_{yz} S$$
$$+ \frac{1}{2}\eta^2 \int \Omega_{yz}^2 S - \frac{1}{6}\Omega_{xz}^3 S \int - \frac{1}{6}\xi^2 \eta \Omega_{xz}^2 \Omega_{yz} S - \frac{1}{2}\frac{\xi}{3}\eta^2 \Omega_{xz}\Omega_{yz}^2 S$$
$$- \frac{1}{6}\eta^3 \Omega_{yz}^3 S$$

Combinants of 2 cubic cubes

$\overline{300}\cdot 0\overline{30} - 3|120\cdot 210|$ — similar to $\Omega_{xz}\Omega_{yz}\Omega_{xz}$ — u_2^3

$-\Omega_{xz} = 3|021\cdot 300| + 6|210\cdot 111| + 3|201\cdot 120|$ — $u_2^2 u_1$

$\frac{1}{2}\Omega_{yz}^2 = 3|300\cdot 012| + 3|102\cdot 210| + 6|111\cdot 201|$ — $u_2 u_1^2$

$\Omega_{xz}\Omega_{yz} = 6[\overline{012}\cdot\overline{210} + \overline{120}\cdot\overline{102} + \overline{201}\cdot 0\overline{21}]$ — $u_0 u_1 u_2$

$-\Omega_{xz} = 3|210\cdot 021| + 6|111\cdot 120| + 3|030\cdot 201|$ — $u_2^2 u_0$

$\frac{1}{2}\Omega_{xz}^2 = 3|120\cdot 012| + 3|102\cdot 030| + 6|021\cdot 111|$ — $u_0^2 u_2$

$3|012\cdot 201| - 3|210\cdot 003| + 6|102\cdot 111|$ — $u_1^2 u_0$

$6|111\cdot 012| + 3|003\cdot 120| + 3|021\cdot 102|$ — $u_0^2 u_1$

$-\frac{1}{6}\Omega_{yz}^3 \quad 00\overline{3}\cdot 3\overline{00} + 3|201\cdot 102|$ — u_1^3

$-\frac{1}{6}\Omega_{xz}^3 \quad 0\overline{30}\cdot 00\overline{3} + 3|012\cdot 021|$ — u_0^3

$-\frac{1}{6}\Omega_{xz}^3 = 030\cdot 003 + 3|012\cdot 021|$ (invalid)

$+\frac{1}{2}\eta^2 \int$

Mechanical Relations

The Coefficients of the Combinations are linear in the forty-two moments of the movements.

We may write the expression $\sum a_j (\alpha x)^j$ in the form $\sum a_j (\alpha x)^j$. We may regard a_j as a point in a forty-two fold linear space, and $(\alpha x)^j$ as representing a point in the same space. $(\alpha x)^j$ or $(\alpha_j)^j$ will be used to denote the point whose coordinates are the coefficients of $(\alpha x)^j$.

The following two Corrections are accounted for by the Examination of the Square Coefficients $(a x)^2 x)^3$ [illegible] Terms.

1

[illegible]

[illegible] $1^2 \cdot 1 \cdot 8 \cdot 1$ or $2 \cdot 1$

[illegible]

[illegible] $1^2 \cdot 3^2 + 1^2 \cdot 4^2$

$+ 1^2 \cdot 2^2 \cdot 3 = 0$.

[illegible]

some Suppose [illegible]

[illegible]

Covariants of $(a_x)^3$

$(a_x)^3$

$(a_x)^2 (a_x)$

$(a_x)(a_x)^2$

$(a_x)^3$

$(a_x)^2 (a_x)$

$(a_x)(a_x)^2$

$(a_x)^3$

$(a_x)^3$ has Five Simple Covariants
$(a_x)^3, (a'_x)^3$.

Here I have made a slight change in my notation, remembering each coefficient from one to ten as before

1 . 300 .

− . 210 .

2 . 120 .

− . 030 .

1 . 051 .

− . 111 .

1 . 021 .

− . 102 .

1 . 012 .

1 . 003 .

We see that in identity 0 the sum $1\bar{2}3\bar{4} + 1\bar{3}\cdot 4\bar{2} + 1\bar{4}\cdot 23 - \ldots$

would be in my original notation

$$\overline{300\cdot 210\cdot 120,080} + \overline{800\cdot 120\cdot 030\cdot 210} + \overline{800\cdot 030\cdot 210\cdot 120}$$

By referring to Table 2 page 101 we see there is only one term which we can get it which is 1 600 by taking 4 of the 2's, the 3, and the 5, giving $2^4 \cdot 3 \cdot 5$. There is no other selection of four primitive terms whose coefficients multiplied together give 1 600 except what. This identity is in the new notation of the form $1\bar{2}\cdot\bar{3}\bar{4} + 1\bar{4}\,2\bar{3} + 1\bar{3}\,4\bar{2}$

There must therefore be some identity between all Covariants whose leading coefficients have this weight

Multiplying the weight of one leading coefficient by the weight of the other leading coefficient we find that $(\alpha\alpha')^2 (a_x)^{n-2} (a'_x)^{n-2}$ and $(b_x)^2 (b'_x)^2$ have leading coefficients with this weight.

Our Syzygy is therefore

$$(\alpha\alpha')^2 (a_x)^{n-2} (a'_x)^{n-2} + \lambda (b_x)^2 (b'_x)^2$$

Or solving for λ we get $\lambda = -\frac{1}{6}$.

$$(\alpha\alpha')^2 (a_x)^{n-2} (a'_x)^{n-2} - \frac{1}{6} (b_x)^2 (b'_x)^2.$$

Or see further identities in just one way, such as a group of four can be selected which will be to a weight 5 31.

1, 2, 3, 5 - added together - or in the old notation

300, 210, 120, 201 give us a weight 5 31. Our identity is therefore, in the form

$1\bar{2}3\bar{5} + 1\bar{3}.5\bar{2} + 1\bar{5}.2\bar{3}$ ...

We have 3 Quadrics whose leading coefficients have a weight 5 31.

The leading coefficients of

$\alpha_{11}\alpha'_{33}(\alpha'x)^3(\alpha x)^2(\alpha'_{23})$.

$|\alpha\beta x|(\alpha\beta)(-x)^3(\beta_{33})$ and $|\alpha\beta x|(\alpha x)^2\beta_{33}$

Substituting for the coefficients of the combinants in the leading coefficients of these three covariants, we find a second syzygy:

$$6[(ab)^2 a_x^2 a_y^2 (xy)^2 (xz)^2 (yz)^2] - [a_x^2 a_y^2 (xy)^4 D_z^2]$$
$$+ [a_x a_y (xy)^2 (ay)^2 (xz)^2 (yz)^2] = 0.$$

This syzygy is identical to the identity $12 \cdot 34 \cdot 56 + 13 \cdot 52 \cdot 46 + 15 \cdot 23 \cdot 46 = 0$.

Covariants of 3 Cubics

3 Cubics have 2 linear combinants. One is a covariant of the sixth degree. The other is a line form of the third order and third class.

The Combinant is

$3\bar{0}0 \cdot 2\bar{1}0 \cdot 2\bar{0}1$ $\quad \Omega_{ba}\Omega_{cb}\Omega_{ca}\Omega_{ba}$ $\quad \Sigma X_0^6$

$2[3\bar{0}0 \cdot 1\bar{2}0 \cdot 2\bar{0}1| + 2\bar{0}0 \cdot 2\bar{1}0 \cdot 1\bar{0}1|]$ $\quad \Sigma X_0^5 X_1$

$3|2\bar{1}0\ 1\bar{2}0\ 2\bar{0}1| + 3\bar{0}0 \cdot 0\bar{3}0\ 2\bar{0}1$

$+ 4|3\bar{0}0 \cdot 1\bar{2}0 \cdot 1\bar{1}1| + 3\bar{0}0 \cdot 2\bar{1}0 \cdot 0\bar{2}1$ $\quad \Sigma X_0^4 X_1^2$

$2[3\bar{0}0 \cdot 0\bar{2}1 \cdot 2\bar{0}1 + 2|3\bar{0}0 \cdot 1\bar{2}0\ 1\bar{0}21$

$+ 3|2\bar{0}1 \cdot 2\bar{1}0 \cdot 1\bar{1}1| + 3\bar{0}0 \cdot 2\bar{1}0 \cdot 0\bar{1}2]$ $\quad \Sigma X_0^4 X_1 X_2$

$2[2\bar{1}0 \cdot 0\bar{3}0 \cdot 2\bar{0}1 + 3|2\bar{1}0 \cdot 1\bar{2}0 \cdot 1\bar{1}1|$

$+ 3\bar{0}0 \cdot 0\bar{3}0 \cdot 1\bar{1}1| + 3\bar{0}0 \cdot 1\bar{2}0 \cdot 0\bar{2}1]$ $\quad \Sigma X_0^3 X_1^3$

$2[3|1\bar{1}1|\bar{2}0 \cdot 2\bar{0}1| + 3|2\bar{1}0\ 0\bar{2}1 \cdot 2\bar{0}1| + 3|2\bar{1}0\ 1\bar{2}0\ 1\bar{0}24$

$+ 3\bar{0}0 \cdot 0\bar{3}0 \cdot 1\bar{0}2 + 3\bar{0}0 \cdot 0\bar{2}1 \cdot 1\bar{1}1 + 2|3\bar{0}0 \cdot 1\bar{2}0 \cdot 0\bar{1}2|]$ $\quad \Sigma X_0^3 X_1^2 X_2$

$9|1\bar{2}0 \cdot 0\bar{1}2 \cdot 2\bar{0}1| + 3|3\bar{0}0 \cdot 0\bar{2}1 \cdot 0\bar{1}2| +$

$3|2\bar{1}0 \cdot 0\bar{2}1 \cdot 1\bar{0}21 + 3|2\bar{0}1 \cdot 0\bar{3}0 \cdot 1\bar{0}21$

$+ 3|2\bar{1}0 \cdot 1\bar{2}0 \cdot 0\bar{0}31 + |3\bar{0}0 \cdot 0\bar{3}0 \cdot 0\bar{0}31$ $\quad X_0 X_1 X_2$

A Curve of the 6th Order

Combinant of 3 cubics

The mixed form is:

$2\bar{0}1 \cdot \bar{1}\bar{1}1 \cdot 0\bar{2}1$	$\Sigma X_2^3 u_2^3$
$3\bar{0}0 \cdot 1\bar{1}1 \cdot 0\bar{2}1 + 2\bar{0}1 \cdot 2\bar{1}0 \cdot 0\bar{2}1 + 2\bar{0}1 \cdot \bar{1}\bar{1}1 \cdot 1\bar{2}0$	$\Sigma X_2^2 X_1$
$3\bar{0}0 \cdot 2\bar{1}0 \cdot 0\bar{2}1 + 3\bar{0}0 \cdot 1\bar{1}1 \cdot 1\bar{2}0 + 2\bar{0}1 \cdot 2\bar{1}0 \cdot 1\bar{2}0$	$\Sigma X_0 X_2^2$
$3\bar{0}0 \cdot 1\bar{2}0 \cdot 0\bar{2}1 + 3\bar{0}0 \cdot 1\bar{1}1 \cdot 0\bar{3}0 + 2\bar{0}1 \cdot 2\bar{1}0 \cdot 0\bar{3}0$	
$+ 2\bar{1}0 \cdot 1\bar{1}1 \cdot 1\bar{2}0$	$X_0 X_1 X_2$
$3\bar{0}0 \cdot 2\bar{1}0 \cdot 1\bar{2}0$	ΣX_0^3
$3\bar{0}0 \cdot 2\bar{1}0 \cdot 0\bar{3}0$	$\Sigma X_0^2 X_1$
$2\|\bar{1}\bar{1}1 \cdot 1\bar{0}2 \cdot 0\bar{2}1\| + 0\bar{1}2 \cdot 2\bar{0}1 \cdot 0\bar{2}1$	$X_2^3 u_2^2 u_0$
$3\|\bar{1}\bar{1}1 \cdot 2\bar{0}1 \cdot 0\bar{2}1\| + 0\bar{1}2 \cdot 3\bar{0}0 \cdot 0\bar{2}1 + 2\|2\bar{1}0 \cdot 1\bar{0}2 \cdot 0\bar{2}1\|$	
$+ 2\|\bar{1}\bar{1}1 \cdot 1\bar{0}2 \cdot 1\bar{2}0\| + 0\bar{1}2 \cdot 2\bar{0}1 \cdot 1\bar{2}0$	$X_0 X_2^2$
$0\bar{1}2 \cdot 2\bar{0}1 \cdot 0\bar{2}1 + 2\|1\bar{2}0 \cdot 1\bar{0}2 \cdot 0\bar{2}1\| + 2\|\bar{1}\bar{1}1 \cdot 1\bar{0}2 \cdot 0\bar{3}0\|$	
$+ 0\bar{1}2 \cdot 2\bar{0}1 \cdot 0\bar{3}0$	$X_1 X_2^2$
$2\|2\bar{1}0 \cdot 2\bar{0}1 \cdot 0\bar{2}1\| + 1\bar{1}1 \cdot 3\bar{0}0 \cdot 0\bar{2}1 + 3\|\bar{1}\bar{1}1 \cdot 2\bar{0}1 \cdot 1\bar{2}0\|$	
$+ 0\bar{1}2 \cdot 3\bar{0}0 \cdot 1\bar{2}0 + 2\|2\bar{1}0 \cdot 1\bar{0}2 \cdot 1\bar{2}0$	$X_0^2 X_2$
$1\bar{2}0 \cdot 2\bar{0}1 \cdot 0\bar{2}1 + 3\|\bar{1}\bar{1}1 \cdot 2\bar{0}1 \cdot 0\bar{3}0\| + 0\bar{1}2 \cdot 3\bar{0}0 \cdot 0\bar{3}0$	
$+ 2\|2\bar{1}0 \cdot 1\bar{0}2 \cdot 0\bar{3}0\| + 0\bar{1}2 \cdot 2\bar{1}0 \cdot 1\bar{2}0 + 2\bar{1}0 \cdot \bar{1}\bar{1}1 \cdot 0\bar{2}1$	
$+ 2\|2\bar{0}1 \cdot \bar{1}\bar{1}1 \cdot 0\bar{3}0\|$	$X_0 X_1 X_2$
$2\|1\bar{2}0 \cdot \bar{1}\bar{1}1 \cdot 0\bar{2}1\| + 0\bar{1}2 \cdot 2\bar{1}0 \cdot 0\bar{3}0 + 2\|1\bar{2}0 \cdot 1\bar{0}2 \cdot 0\bar{3}0\|$	
$+ 0\bar{2}1 \cdot 2\bar{0}1 \cdot 0\bar{3}0$	$X_1^2 X_2$

Coordinate Ring of Cubic Surface - Continued

$2|\bar{2}10\cdot 20\bar{1}\cdot 1\bar{2}0| + 111\cdot 300\cdot 120$ $\quad x_0^2 x_2$

$2|2\bar{1}0\cdot 20\bar{1}\cdot 0\bar{3}0| + 111\cdot 300\cdot 030$ $\quad x_0^2 x_1$,

$+120\cdot 300\cdot 021 + 2\bar{1}0\cdot 111\cdot 120$ $\quad x_0 x_1$,,

$2|120\cdot 20\bar{1}\cdot 0\bar{3}0 + 021\cdot 300\cdot 030 + 2\bar{1}0\cdot 120\cdot 021$

$+2\bar{1}0\ 111\cdot 030$ $\quad x_0 x_1^2$

$2|120\ 111\cdot 0\bar{3}01 + 021\cdot 2\bar{1}0\cdot 030$ $\quad x_1^3$

$+|10\bar{2}\cdot 111\cdot 0\bar{1}21 + 201\cdot 003\ 021$ $\quad \leq x_2^3 - 3x_1x_2$

$4|10\bar{2}\cdot 20\bar{1}\cdot 021| + 4|20\bar{1}\cdot 111\cdot 0121 + 3\bar{0}0\cdot 003\cdot 021|$

$+4|102\cdot 2\bar{1}0\cdot 012| + 20\bar{1}\cdot 003\cdot 120$ $\quad \leq x_0 x_2^2$,,

$4|\bar{2}01\cdot 111\cdot 021| + 3|\bar{2}01\ 120\cdot 01\bar{2}| + 20\bar{1}\cdot 10\bar{2}\cdot 030$

$+3\bar{0}0\cdot 0\bar{0}3\cdot 0\bar{3}0 + 3\bar{0}\bar{0}\cdot 01\bar{2}\cdot 0\bar{2}1 + 3|\bar{1}02\cdot 2\bar{1}0\cdot 02\bar{1}|$

$+2\bar{1}0\cdot 00\bar{3}\cdot 1\bar{2}0 + 4|111\cdot 102\cdot 120| + 4|111\ 2\bar{1}0\cdot 012|$ $\quad x_0 x_1 x_2$,,

As previously stated

3 cubics have 9 common

One of the cubics

[illegible] and third lines

(Covariants of four cubics

The Covariant Theta four binary cubics
$(a x)^3, (b x)^3 (c x)^3 (d x)^3$ is found to be dependent to:
$(ab)^2 (ac)^2 (ad)^2 (bc)^2 (bd)^2 (cd)^2$.

From this we get the following Covariant of four binary Cubics.
We find by trial that
$3\bar{1}0.2\bar{1}0.1\bar{2}0.0\bar{3}0$ have the annihilators $\Delta_0, \Delta_1, \Delta_2, \Delta_3$.
$3\bar{0}0.2\bar{1}0.1\bar{2}0.0\bar{3}0$ to the [illegible] the Coefficients A_x a_x^6.

The Covariant [illegible] the following page.

$3\bar{0}0 \cdot \bar{2}10 \cdot 120 \cdot 0\bar{3}0$ $\quad \sum u_0^6$

$\frac{1}{3} L_{ae}^3 = 3|\bar{2}10 \cdot 210 \cdot 120 \cdot 0\bar{3}0| + 2|111 \cdot \bar{3}00 \cdot 120 \cdot 0\bar{3}0| + \bar{2}10\ \bar{3}00\ 021\ 030$ $\quad \sum u_0^2 u_2^3$

$L_{ae} L_{ae}^2 = 3|120 \cdot 021 \cdot \bar{2}10 \cdot \bar{3}00| + 2|111\ 030\ \bar{2}10\ \bar{3}00| + |120 \cdot 030 \cdot \bar{2}01\ \bar{3}00|$ $\quad \sum u_1 u_2^3$

$\frac{1}{2} L_{ae}^2 L_{ae}^2 = 6|111 \cdot \bar{2}01 \cdot 120 \cdot 0\bar{3}0| + 3|\bar{2}10 \cdot \bar{1}02\ 120\ 030| + 3|\bar{2}10 \cdot \bar{2}01 \cdot 021\ 03\bar{1}|$

$+ \quad + 0\bar{1}2 \cdot \bar{3}00 \cdot 120 \cdot 030 + 2|111\ 300\ 021 \cdot 030|$ $\quad \sum u_0^2 u_2^4$

$L_{ae} L_{ae}^3 = 6|\bar{2}10 \cdot 2\bar{1} \cdot 111 \cdot 0\bar{3}0| + 9|2\bar{1}0 \cdot \bar{2}01 \cdot 120 \cdot 0\bar{2}1| + 2|10\bar{2}\ 300$ $\quad \sum u_0 u_2^4$

$+ 6|111\ 300 \cdot 120 \cdot 0\bar{2}1| + \bar{2}01 \cdot 300 \cdot 021 \cdot 030 + |\bar{2}10\ 300 \cdot 0\bar{1}\bar{2} \cdot 03\bar{1}|$

$\frac{1}{2} L_{ae}^4 = 6|111 \cdot 021 \cdot 2\bar{1}0 \cdot 300| + 3|120\ 012 \cdot \bar{2}10\ 300| + 3|120\ 021\ 201 \cdot 300|$

$+ \bar{1}02 \cdot 030 \cdot \bar{2}10 \cdot 300 + 2|111 \cdot 030 \cdot 201 \cdot \bar{3}00|$ $\quad \sum u_1^2 u_2^4$

$\frac{1}{6} L_{ae}^3 = 3|\bar{2}01 \cdot 012 \cdot 1\bar{1}0\ 030| + 6|102\ 111 \cdot 1\bar{1}0\ 030| + 6|\bar{2}01 \cdot 111 \cdot 021 \cdot 030|$

$+ 003 \cdot \bar{2}10 \cdot 120 \cdot 030 + 3|102\ \bar{2}10 \cdot 021 \cdot 030| + 300 \cdot 012 \cdot 021 \cdot 030$ $\quad \sum u_0^3 u_2^3$

$\frac{1}{2} L_{ae}^2 L_{ae}^2 = 9|\bar{2}01 \cdot 102\ 120 \cdot 030| + 18|\bar{2}01 \cdot 111\ 120 \cdot 021| + 3|102 \cdot \bar{2}10\ \overset{120}{\hat{}}\ 030|$

$+ 6|102 \cdot \bar{2}10 \cdot 111 \cdot 03\bar{1}| + 9|102 \cdot \bar{2}10\ 120 \cdot 021| + 6|\bar{2}01 \cdot \bar{2}10\ 012 \cdot 030|$

$+ 2|3\bar{0}0 \cdot 111 \cdot 0\bar{1}2 \cdot 0\bar{3}0| + 3|3\bar{0}0 \cdot 0\bar{1}2 \cdot 120 \cdot 021| + 4|300\ 102 \cdot 021 \cdot 030|$

$+ 3\bar{0}0 \cdot 003 \cdot \bar{1}20 \cdot 0\bar{3}0$ $\quad \sum u_0^2 u_1 u_2^2$

$\frac{1}{4} L_{ae}^2 L_{ae}^2 = 9|\bar{2}01 \cdot 102\ 120\ 0\bar{2}1| + 18|102 \cdot \bar{2}10\ 111\ 021| + 9|102 \cdot \bar{2}10 \cdot 120 \cdot 012|$

$+ 3|\bar{2}01\ \bar{2}10\ 012\ 0\bar{3}1| + 4|\bar{3}00 \cdot 102 \cdot 012 \cdot 0\bar{3}0| + 3\bar{0}0 \cdot 111 \cdot 003 \cdot 0\bar{3}0$

$+ 3|300\ 003\ 120 \cdot 0\bar{2}1| + 300 \cdot 003 \cdot 111\ 030$ $\quad \sum u_0^2 u_1^2 u_2$

Alternate 17 + 3 Antecedents.

A further amusement of four cubes is one of the fifth order and second class.

$\overline{201}\cdot\overline{111}\cdot\overline{021}\cdot\overline{003}$ is annihilated by Ω_{ab} and Ω_{ai}.

$\overline{201}\cdot\overline{111}\cdot\overline{021}\cdot\overline{003}$ is therefore the leading coefficient of $x_2^i x_3^j$ where i and j are to be determined.

Expanding our Covariant operator we see that Ω_{ab} operating on x_2^i gives us the the coefficient of $x_2^{i-1} x_1$. $\frac{1}{2}\Omega_{ab}^2$ gives the coefficient of $x_2^{i-2} x_1^2$. In order to find how many times the variable x_1 occurs operate with Ω_{ab} j times. This can be done

five times. The Covariant is, therefore of the fifth order

The Class of the Covariant is found in a similar manner: After finding the Coefficient of u_2 the Operator to be used is

$$u_2^{\beta}\left(e^{-\frac{u_0}{u_2}\left(\Omega_{ae} - X_0 \frac{\partial}{\partial X_2}\right) + \frac{u_1}{u_2}\left(-2ae - X_1 \frac{\partial}{\partial X_2}\right)}\right) \beta.$$

Expanding this we get

$$u_2^{\beta}\beta + u_2^{\beta-1} u_0 \left(-\Omega_{ae} + X_0 \frac{\partial}{\partial X_2}\right) + \cdots$$

If we have the term u_2^{β} operating with $\left(-\Omega_{ae} + X_0 \frac{\partial}{\partial X_2}\right)$ gives the coefficient of term $u_2^{\beta-1} u_0$. u_0 occurs one more time as we can operate with $\left(-\Omega_{ae} + X_0 \frac{\partial}{\partial X_2}\right)$ upon the coefficient of $X_2^{\beta} u_2^{\beta}$. By operating twice on $201.111\ 021\ 003$ the coefficient reduces to $4|10.2.012.021.003$. Any further operation causes it to vanish.

McCoun's Theorem of the Seven Class.

Combinant of 4 Particles

$2\bar{0}1.\bar{1}11.0\bar{2}1.00\bar{3}$. annihilated by Ω_{ab} and Ω_{ca} | $x_2^5 \cdot \frac{u_2^2}{2}$

$\frac{u}{2} = 2\bar{1}0\ \bar{1}11.0\bar{2}1.00\bar{3} + \bar{1}01\ 1\bar{1}0\ 0\bar{2}1.00\bar{3} + \bar{2}11\ \bar{1}11\ 030\ 00\bar{3} + 3|201\ \bar{1}11\ 02\bar{1}\ 00\bar{3}|$ | $x_1 x_2^4$

$\frac{1}{2}\Omega_{ca}^2 = 300\ 111\ 0\bar{2}1\ 00\bar{3} + \bar{1}01\ \bar{2}10\ 021\ 00\bar{3} + \bar{1}01\ 111\ \bar{1}20\ 00\bar{3} + 3|\bar{1}01\ 111\ 021\ 10\bar{2}|$ | $x_0^1 x_2^4$

$\frac{1}{2}\Omega_{ab}^2 = 2\bar{1}0\ 1\bar{2}0\ 0\bar{2}1\ 00\bar{3} + 2\bar{1}0\ \bar{1}11\ 030\ 00\bar{3} + 3|2\bar{1}0\ \bar{1}11\ 0\bar{2}1\ 01\bar{2}|$

$+ \bar{2}01\ 1\bar{2}0\ 0\bar{3}0\ 00\bar{3} + 3|\bar{2}01\ 1\bar{1}0\ 0\bar{2}1\ 01\bar{2}| + 3|\bar{2}01\ 111\ 0\bar{3}0\ \underset{012}{\bar{1}\bar{2}1}|$ | $\sum x_1^2 x_2^3$

$\frac{1}{2}\Omega_{ca}^2\Omega_{ab} = 2\bar{1}0\ \bar{1}11\ 1\bar{2}0\ 00\bar{3} + 3|2\bar{1}0\ \bar{1}11\ 0\bar{2}1\ 10\bar{2}| + 300\ 1\bar{2}0\ 0\bar{2}1\ 00\bar{3} + 3|\bar{2}01\ 1\bar{2}0\ 0\bar{2}1\ 10\bar{2}|$

$+ 3\bar{0}0\ \bar{1}11\ 030\ 00\bar{3} + \bar{2}01\ 2\bar{1}0\ 0\bar{3}0\ 00\bar{3} + 3|\bar{2}01\ \bar{1}11\ 030\ 10\bar{2}|$

$+ 3|3\bar{0}0\ \bar{1}11\ 0\bar{2}1\ 01\bar{2}| + 3|\bar{2}01\ 2\bar{1}0\ 0\bar{2}1\ 01\bar{2}| + 3|\bar{2}01\ \bar{1}11\ \underset{110}{\bar{1}\bar{2}0}\ 01\bar{2}|$ | $\sum x_0 x_1 x_2^3$

$\frac{1}{6}\Omega_{ab}^3 = 2\bar{1}0\ 1\bar{2}0\ 0\bar{3}0\ 00\bar{3} + 3|2\bar{1}0\ 1\bar{2}0\ 0\bar{2}1\ 01\bar{2}| + 3|2\bar{1}0\ \bar{1}11\ 0\bar{3}0\ 01\bar{2}|$

$+ 3|\bar{2}01\ 1\bar{2}0\ 0\bar{3}0\ 01\bar{2}| + 2|\bar{2}01\ 111\ 0\bar{3}0\ 0\bar{2}1|$ | $\sum x_1^3 x_2^3$

$\frac{1}{24}\Omega_{ca}^4 = 3|2\bar{1}0\ 1\bar{2}0\ 0\bar{3}0\ 01\bar{2}| + 2|2\bar{1}0\ \bar{1}11\ 0\bar{3}0\ 0\bar{2}1|$

$+ 2|\bar{2}01\ 1\bar{2}0\ 0\bar{3}0\ 0\bar{2}1|$ | $\sum x_1^4 x_2$

$\frac{1}{120}\Omega_{ab}^5 = 2\bar{1}0\ 1\bar{1}0\ 0\bar{3}0\ 0\bar{2}1$ | $\sum x_1^5$

$\frac{1}{2}\Omega_{ca}^2 = 3\bar{0}0\ 2\bar{1}0\ 0\bar{2}1\ 00\bar{3} + 3\bar{0}0\ \bar{1}11\ 1\bar{2}0\ 00\bar{3}$

$+ 3|3\bar{0}0\ \bar{1}11\ 0\bar{2}1\ 10\bar{2}| + 2\bar{0}1\ 2\bar{1}0\ 1\bar{2}0\ 00\bar{3}$

$+ 3|\bar{2}01\ 2\bar{1}0\ 0\bar{2}1\ 10\bar{2}| + 3|2\bar{0}1\ 1\bar{1}1\ 1\bar{2}0\ 10\bar{2}|$ | $x_0^3 x_2^3$

Fifth Order and Second Class

Biography

Josiah Wesley Hall was born October 9, 1877 in Jamestown, Kansas. His early training was received in the public schools and in Kansas-College State Normal School, from which he graduated in 1903.

He was Elected Principal of the High School at Emporia in 1904, and was re-elected for three years.

In the summer of 1905 he entered the University of Chicago, where in 1910 he took the degree Bachelor of Science. In 1911 he entered Johns Hopkins University as a graduate student in Mathematics, Living Education and astronomy as major and secondary subjects.